(1) Решить ур-е.

$z^6+2z^3+1=0$

Замена $z^3=t$

$t^2+2t+1=0$

$t=-1 \Rightarrow z^3=-1$

$z_k=\sqrt[n]{|z|}\left(\cos\frac{\arg z+2\pi k}{n}+i\sin\frac{\arg z+2\pi k}{n}\right)$

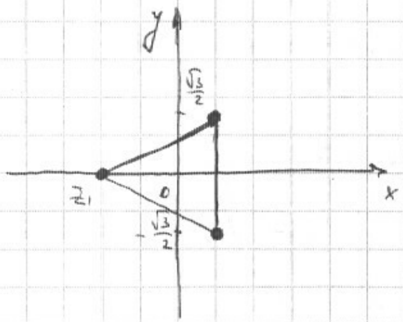
$|z|=|-1|=1$

$\arg z=\arg(-1)=\pi$

$z_0=1\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\frac{1}{2}+i\frac{\sqrt{3}}{2}$

$z_1=1\left(\cos\pi+i\sin\pi\right)=-1+i\cdot 0=-1$

$z_2=1\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)=\frac{1}{2}-i\frac{\sqrt{3}}{2}$



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② При каких значениях константы a ф-ция $u(x,y)$ или $v(x,y)$ явл. вещественной или соответственно мнимой частью ф-ции $f(z)$? Восстановить $f(z)$.

$v(x,y) = e^{-y}(x\cos x - y\sin ax)$

$(-e^{-y}(x\cos x - y\sin ax) + e^{-y}(-\sin ax))'_y +$

$+ e^{-y}(-2\sin x - x\cos x + a^2 y\sin ax) = 0$

$e^{-y}(x\cos x - y\sin ax + 2\sin ax) + e^{-y}(-2\sin x -$

$- x\cos x + a^2 y\sin ax) = 0$

$\underline{a = 1}$

$v(x,y) = e^{-y}(x\cos x - y\sin x)$

$u(x,y) = \int\limits_{(x_0,y_0)}^{(x,y)} (e^{-y}(-x\cos x + y\sin x - \sin x)\,dx -$

$- e^{-y}(\cos x - x\sin x - y\cos x)\,dy) =$

$= \int\limits_{x_0}^{x} e^{-y_0}(-x\cos x + y_0\sin x - \sin x)\,dx - \int\limits_{y_0}^{y} e^{-y}(\cos x -$

$- x\sin x - y\cos x)\,dy = -e^{-y_0}\left(\int\limits_{x_0}^{x} x\cos x\,dx - y_0\int\limits_{x_0}^{x}\sin x\,dx +\right.$

$\left.+ \int\limits_{x_0}^{x}\sin x\,dx\right) - \cos x\int\limits_{y_0}^{y} e^{-y}dy + x\sin x\int\limits_{y_0}^{y} e^{-y} + \cos x\int\limits_{y_0}^{y} e^{-y}y\,dy =$

$= -e^{-y_0}(x\sin x - x_0\sin x_0 + \cos x - \cos x_0 + y_0\cos x -$

$- y_0\cos x_0 - \cos x + \cos x_0) + \cos x\, e^{-y} - \cos x\, e^{-y_0} - x\sin x\, e^{-y} +$

$+ x\sin x\, e^{-y_0} - \cos x\,(e^{-y}y - e^{-y_0}y_0 + e^{-y} - e^{-y_0}) =$

$= -e^{-y_0}x\sin x - e^{-y_0}y_0\cos x + \cancel{\cos x\, e^{-y}} - \cancel{\cos x\, e^{-y_0}} -$

$- x\sin x\, e^{-y} + x\cancel{\sin x\, e^{-y_0}} - \cos x\, e^{-y}y + \cancel{\cos x\, e^{-y_0}} -$

$- \cancel{\cos x\, e^{-y}} + \cancel{\cos x\, e^{-y_0}} + C = -e^{-y}(x\sin x + y\cos x) + C$

$f(z) = -e^{-y}(x\sin x + y\cos x) + C + ie^{-y}(x\cos x - y\sin x) =$

$= -e^{-y}x\sin x - \underline{e^{-y}y\cos x} + ie^{-y}x\cos x - \underline{ie^{-y}y\sin x} + C =$

$= -ye^{-y}(\cos x + i\sin x) + xie^{-y}(\cos x + i\sin x) =$

$= e^{-y+ix}(-y + ix) + C = ie^{iz}z + C$ ⊕

$(e^{-y}(x\cos x - y\sin ax))''_x + (e^{-y}(x\cos x - y\sin ax))''_y = 0.$

$(e^{-y}(\cos x - x\sin x - ay\cos ax))'_x + (-e^{-y}(x\cos x - y\sin ax) +$

$+ e^{-y}(-\sin ax))'_y = 0.$

$e^{-y}(-2\sin x - x\cos x + a^2 y\sin ax) + e^{-y}(x\cos x + 2\sin ax - y\sin ax) =$

$= 0$

$e^{-y}(-2\sin x - \cancel{x\cos x} + a^2 y\sin ax + \cancel{x\cos x} + 2\sin ax - y\sin x) = 0$

$-2\sin x + a^2 y\sin ax + 2\sin ax - y\sin ax = 0.$

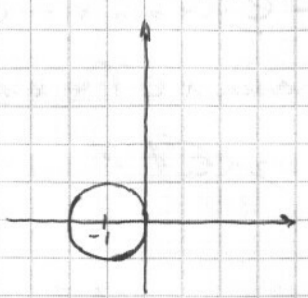
$\sin x = \sin ax$

$a = 1$

③ Ряд Лорана, найти $\operatorname*{res}_{z=z_0} f(z)$

$$f(z)=\frac{z-1}{z(z+1)}, \quad z_0=-1$$

$$f(z)=\frac{A}{z}+\frac{B}{z+1}=-\frac{1}{z}+\frac{2}{z+1}$$



$$A(z+1)+Bz=z-1$$

$$\begin{cases} A+B=1 \\ A=-1 \end{cases} \Rightarrow B=2$$

$$f(z)=\frac{2}{z+1}-\frac{1}{z}$$

$D_1:\ 0<|z+1|<1$

$D_2:\ |z+1|>1$

1) $D_1:\ -\frac{1}{z}=\frac{1}{1-(z+1)}=\frac{1}{1-a}=\sum_{n=0}^{\infty} a^n=$

$=\sum_{n=0}^{\infty}(z+1)^n \quad (|z+1|=|a| \quad |a|<1 \text{ в обл. } D_1)$

$$f(z)=\frac{2}{z+1}+\sum_{n=0}^{\infty}(z+1)^n$$

2) $D_2:\ \frac{2}{z+1}+\frac{1}{\left(\left(\frac{1}{z+1}\right)-1\right)(z+1)}=$

$$=\frac{2}{z+1}-\frac{1}{1-\frac{1}{z+1}}\cdot\frac{1}{z+1}=\frac{2}{z+1}-\frac{1}{z+1}\sum_{n=0}^{\infty}\frac{1}{(z+1)^n}=$$

$$=\frac{2}{z+1}-\left(\frac{1}{z+1}+\frac{1}{(z+1)^2}+\dots\right)=\frac{1}{z+1}-\frac{1}{(z+1)^2}-\frac{1}{(z+1)^3}=$$

$$=\frac{1}{z+1}-\sum_{n=2}^{\infty}\frac{1}{(z+1)^n}.$$

Рассм. точки $z_0=0,\ z_1=-1$

$z_0:\ f(z)=\dfrac{\frac{z-1}{z+1}}{z}=\dfrac{u(z)}{\psi(z)}$

$u(z) \neq 0$, $\psi(z) = 0$

$\Rightarrow$ z_0 - полюс 1-го пор-ка

$$z_1: f(z) = \frac{\frac{-z-1}{z}}{z+1} = \frac{u(z)}{\psi(z)}$$

$u(z) \neq 0$, $\psi(z) = 0$

$\Rightarrow$ $z_1 = -1$ - полюс 1-го пор-ка

$$\operatorname*{res}_{z_0=0} f(z) = \lim_{z\to 0} z \cdot \frac{-z-1}{z(z+1)} = \lim_{z \to 0} \frac{-z-1}{z+1} = -1$$

$$\operatorname*{res}_{z_0=-1} f(z) = \cancel{\lim_{z\to -1} (z+1) \cdot \frac{-z-1}{z(z+1)} = \lim_{z\to -1} \frac{-z-1}{z}}$$

$$= \cancel{\frac{1-1}{-1} = 0} = \frac{z-1}{z(z+1)'}\Big|_{z=-1} = \frac{z-1}{z}\Big|_{z=-1} = \frac{-1-1}{-1} = 2$$

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4 Ряд Лорана, найти $\operatorname{res}_{z_0} f(z)$

$$f(z) = z \cdot \cos\frac{1}{z-2}, \quad z_0 = 2$$

$$z \cdot \cos\frac{1}{z-2} = (z-2+2) \cdot \cos\frac{1}{z-2} =$$

$$= (z-2) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-2)^n (2n)!} + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-2)^n \cdot 2n!} =$$

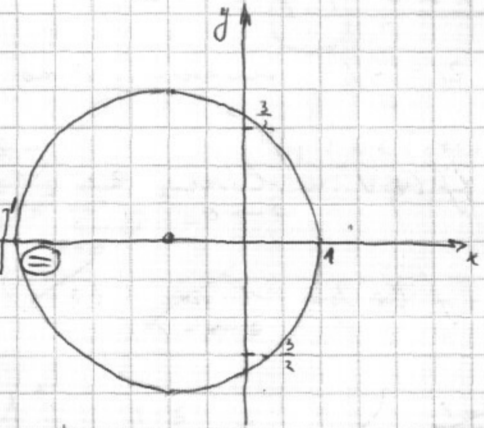
$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(z-2)^{n-1} \cdot 2n!} + \frac{2(-1)^n}{(z-2)^n \cdot (2n)!} \right).$$

c_n - ?

5) Вычислить интеграл по замкн. контуру с помощью вычетов.

$$f(z) = \frac{z^2+1}{(2z+3)z^2}, \quad |z+1| = 2$$

$$\oint\limits_{|z+1|=2} \frac{z^2+1}{(2z+3)z^2}\, dz = 2\pi i \sum_{k=0}^{1} \operatorname*{res}_{z=z_k} f(z)$$



$z=0$ — полюс 2-го пор-ка

$z=-\frac{3}{2}$ — полюс 1-го пор-ка

$$\operatorname*{res}_{z=0} f(z) = \lim_{z\to 0}\left[z^2\frac{z^2+1}{(2z+3)z^2}\right]' = \lim_{z\to 0}\left(\frac{z^2+1}{2z+3}\right)' =$$

$$= \lim_{z\to 0}\frac{2z(2z+3)-(z^2+1)\cdot 2}{(2z+3)^2} = \frac{0-2}{3^2} = -\frac{2}{9}$$

$$\operatorname*{res}_{z=-3/2} f(z) = \lim_{z\to -3/2}\left(z+\frac{3}{2}\right)\frac{z^2+1}{(2z+3)z^2} =$$

$$= \lim_{z\to -3/2}\frac{z^2+1}{2z^2} \,?\, \cancel{\lim_{z\to -3/2}\frac{z^3}{2}+\frac{1}{2z} = \frac{1}{2}} + \frac{1}{2\left(-\frac{3}{2}\right)^2} =$$

$$= \cancel{\frac{1}{2} + \frac{1}{2\frac{9}{4}} = \frac{1}{2}} + \frac{2}{9} = \frac{\frac{9}{4}+1}{\frac{9}{2}} = \frac{1}{2} + \frac{2}{9}$$

$$\oint\limits_{0} f(z) = 2\pi i\left(\frac{1}{2} + \frac{2}{9} - \frac{2}{9}\right) = \pi i$$

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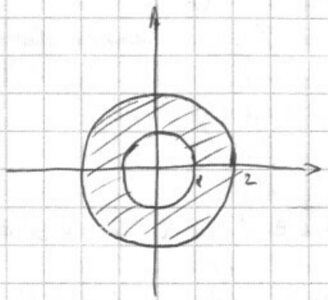
⑦ Теорема Руше

$z^5 - 5z^2 + 2z + 1 = 0$, $1 < |z| < 2$

$D: |z| < 2$, $\Gamma: |z| = 2$



$f(z) = z^5$

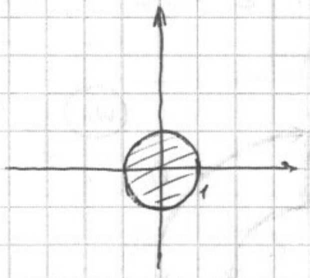
$g(z) = -5z^2 + 2z + 1$

$\forall z \in \Gamma: \; |f(z)| = |z|^5 = 32$

$|g(z)| = |-5z^2 + 2z + 1| \leq 5|z|^2 + 2|z| + 1 = 20 + 4 + 1 = 25$

$\Rightarrow \; |f(z)| > |g(z)|$

$\Rightarrow$ в обл. D $N_F = N_f = 5$



$D_1: |z| < 1$, $\Gamma_1: |z| = 1$

$f(z) = -5z^2$

$g(z) = z^5 + 2z + 1$

$\forall z \in \Gamma_1: \; |f(z)| = |-5z^2| = 5$

$|g(z)| = |z^5 + 2z + 1| \leq |z|^5 + 2|z| + 1 = 1 + 2 + 1 = 4$

$\Rightarrow \; |f(z)| > |g(z)|$

$\Rightarrow$ в обл. D_1 $N_F = N_f = 2$

Тогда, в кольце $1 < |z| < 2$ $N_F = 5 - 2 = 3$

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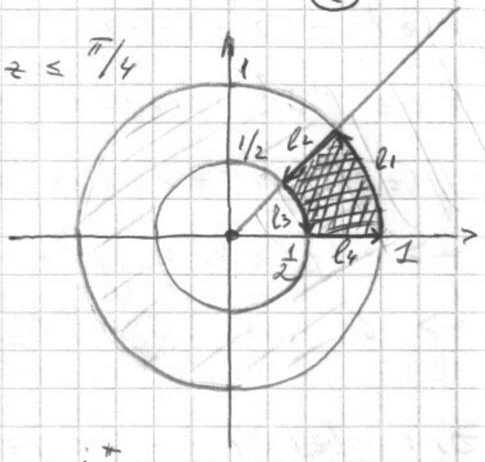
⑨ Определить мн-во D_2 пл-ти W, на кот. отображ. мн-во D_1 пл-ти z зад. ф-цией $w = f(z)$.

$f(z) = (\sqrt{3} + i)z^2 + 1 + 5i$

$D_1: \ 1/2 < |z| < 1, \quad 0 \le \arg z \le \pi/4$



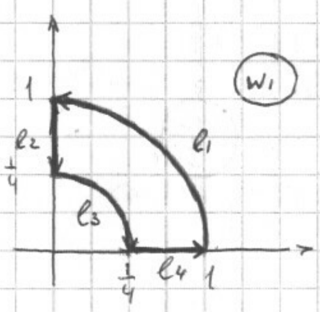
$l_1: re^{i\varphi}, \ 0 < \varphi < \frac{\pi}{4}$

$r = \sqrt{x^2 + y^2}$

$\varphi = \arg z$

$w_1 = z^2$

$l_1: \ z^2 = 1e^{2i\varphi}, \ 0 < \varphi < \frac{\pi}{4}$



$l_2: z = re^{i\frac{\pi}{4}}$

$w_1 = l_2: \ z^2 = r^2 e^{2i\frac{\pi}{4}}, \ 1 > r > \frac{1}{2}$

$l_3: z = \frac{1}{2}e^{i\varphi}, \ \frac{\pi}{4} > \varphi > 0$

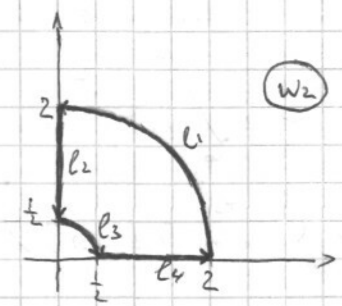
$z^2 = \frac{1}{4}e^{2i\varphi}, \ \frac{\pi}{4} > \varphi > 0$

$l_4: re^{i\cdot 0}, \ \frac{1}{2} < r < 1$

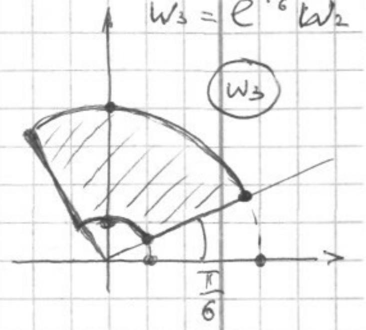
$w = z^2 \ \ l_4: \ r^2 e^{2i\cdot 0}, \ \frac{1}{2} < r < 1$

$w_2 = |a|\, w_1$ $|a| = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$l_1 = 2e^{z}$

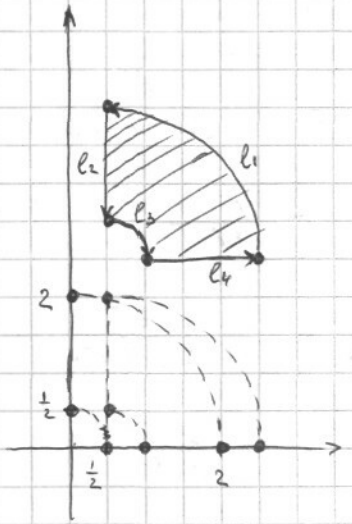


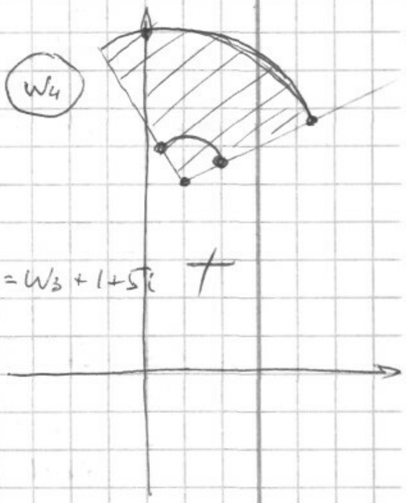
$w_3 = e^{i\frac{\pi}{6}} w_2$



$w_3 = w_2 + b$

$b = 1 + 5i$





$w_4 = w_3 + 1 + 5i$

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⑥ Вычислить несобств. интеграл $\int_a^b f(x)\,dx$ с помощью вычетов

$$\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+9)}\,dx =$$

$$= 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{res}_{z=z_k} \left(\frac{z^2}{(z^2+1)(z^2+9)} \right)$$

$z^2+1=0$ $\qquad$ $z^2+9=0$

$z=\sqrt{-1}=i$ $\qquad$ $z=3i$

$$\operatorname*{res}_{z=i} \frac{z^2}{(z^2+1)(z^2+9)} = \frac{z^2}{2z\,(z^2+9)}\Bigg|_{z=i} =$$

$$= \frac{-1}{2i\,(-1+9)} = -\frac{1}{16i} = \frac{1}{16}i$$

$$\operatorname*{res}_{z=3i} \frac{z^2}{(z^2+1)(z^2+9)} = \frac{z^2}{2z\,(z^2+1)}\Bigg|_{z=3i} =$$

$$= \frac{-9}{6i\cdot(-8)} = -\frac{3}{16}i$$

$$= \pi i \left(\frac{1}{16}i - \frac{3}{16}i \right) = \frac{\pi}{8}.$$

⊕

(4) $f(z) = z \cdot \cos \frac{1}{z-2}$, $z_0 = 2$

$$z \cdot \cos \frac{1}{z-2} = (z-2+2) \cdot \cos \frac{1}{z-2} \Leftrightarrow$$

$$= \cancel{(z-2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-2)^{2n-1} (2n)!} + \sum_{n=0}^{\infty} 2 (}$$

$$\Leftrightarrow (z-2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-2)^{2n} (2n)!} + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-2)^{2n} (2n)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(z-2)^{2n-1} (2n)!} + \sum_{n=0}^{\infty} \frac{2(-1)^n}{(z-2)^{2n} (2n)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{(z-2)^{2n-1}} + \frac{2}{(z-2)^{2n}} \right).$$

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